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Infinite Systems of Nonlinear Oscillation Equations

R. W. DICKEY*

Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706

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1. INTRODUCTION

The purpose of this paper is to discuss the existence and uniqueness of solutions to infinite systems of nonlinear oscillation equations of the form

$$T_j'' + \left(a_j + b_j \sum_{l=1}^{\infty} b_l T_l^2\right) T_j = 0, \qquad j = 1, 2,..., \infty \tag{1.1}$$

(the prime indicates differentiation with respect to t), with initial conditions

$$T_j(0) = \alpha_j\,, \tag{1.2a}$$

$$T_j'(0) = \beta_j\,. \tag{1.2b}$$

Equivalently, the system (1.1) can be written in the form

$$\mathbf{T}'' + (A + (\mathbf{T}^* B \mathbf{T})\, B)\, \mathbf{T} = 0, \tag{1.3}$$

where $\mathbf{T}$ is an infinite dimensional column vector with components T_j and $\mathbf{T}^*$ is the transpose of $\mathbf{T}$. The matrices A and B are given by

$$A = \operatorname{diag}(a_1\,, a_2\,,...) \tag{1.4a}$$

$$B = \operatorname{diag}(b_1\,, b_2\,,...). \tag{1.4b}$$

Note that if A and B are not diagonal but can be simultaneously diagonalized by an orthogonal transformation, then there is no loss of generality in assuming A and B are diagonal.

In what follows, certain conditions will be imposed on the terms a_j and b_j occuring in (1.1). In particular, it will be assumed that

$$b_j > 0, \qquad j = 1, 2,..., \infty, \tag{1.5}$$

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i.e., B is positive definite, and secondly it will be assumed that there exists a constant $\mu > 0$ such that

$$a_j \geqslant \mu b_j, \qquad j = 1, 2, \ldots, \infty. \tag{1.6}$$

Note that no assumption is made on the boundedness of either a_j or b_j. In fact, situations arise in the applications in which a_j and b_j are unbounded. For example, it is shown in [1] that the existence and uniqueness of a Fourier series solution to the nonlinear partial integro-differential equation describing the dynamic behavior of an "extensible" string reduces to a discussion of (1.1) with $a_j = k_0 j^2$ and $b_j = k_1 j^2$ ($k_0 \geqslant 0$, $k_1 > 0$). Similarly, in the treatment of nonlinear beams (cf. [2]) equations of the form (1.1) arise with $a_j = C_0 j^4 + C_1 j^2$ and $b_j = C_2 j^2$. If the beam is in tension, then C_0, C_1, and C_2 are positive constants.

It is of interest to note the relation between the system (1.1) and the scalar Duffing equation (cf. [3]). Eq. (1.1) has a solution $T_j = 0$ for $j \neq k$ if T_k is a solution of

$$T_k'' + a_k T_k + b_k^2 T_k^3 = 0, \tag{1.7}$$

i.e., if T_k is a solution of Duffing's equation. Thus, in a sense, the system (1.1) is a generalization of the scalar Duffing equation to an infinite dimensional system. This feature has been noted in the particular case of the nonlinear beam by Woinowsky-Krieger [4] and Eisley [5], as well as others.

In Section 2 of this paper, the existence of solutions to (1.1) will be proven, and in Section 3 the uniqueness of these solutions will be shown.

2. Existence

It is convenient to begin the discussion of the existence of solutions to (1.1) by treating related finite systems. Define functions $T_{j,N}$ to be the solution of

$$T_{j,N}'' + \left(a_j + b_j \sum_{l=1}^{N} b_l T_{l,N}^2\right) T_{j,N} = 0, \qquad j = 1, 2, \ldots, N, \tag{2.1}$$

satisfying the initial conditions (1.2) for $j \leqslant N$. There is no difficulty in using the method of successive approximation (cf. [6]) to prove the existence of a (unique) solution to (2.1) in some interval $0 \leqslant t < t_1$. In order to show that the solution of (2.1) actually exists for all $t \geqslant 0$, it is sufficient to show that $|T_{j,N}|$ and $|T_{j,N}'|$ are bounded for all $t \geqslant 0$ (cf. [6]). However, if (2.1) is multiplied by $T_{j,N}$, summed over j, and integrated, it follows that

$$\sum_{j=1}^{N} (T_{j,N}')^2 + \sum_{j=1}^{N} a_j T_{j,N}^2 + \frac{1}{2}\left(\sum_{j=1}^{N} b_j T_{j,N}^2\right)^2 = h_N, \tag{2.2}$$

where

$$h_N = \sum_{j=1}^{N} \beta_j^2 + \sum_{j=1}^{N} a_j\alpha_j^2 + \frac{1}{2}\Big(\sum_{j=1}^{N} b_j\alpha_j^2\Big)^2. \tag{2.3}$$

It is an immediate consequence of (2.3) that $| T_{j,N} |$ and $| T'_{j,N} |$ are bounded.

The object in what follows will be to show that the functions $T'_{j,N}$ converge to solutions of (1.1) as $N \to \infty$. For ease of notation, extend the definition of $T_{j,N}$ for $j > N$ by defining $T_{j,N} \equiv 0$ for $j > N$. Combining this definition with (2.1) it follows that $T_{j,N}$ is a solution of (1.1) satisfying the initial conditions (1.2) for $j \leqslant N$ and the initial conditions $T_{j,N}(0) = T_{j,N}(0) = 0$ for $j > N$, i.e., $T_{j,N}$, is a solution of

$$T''_{j,N} + (a_j + b_j\Delta_N)\, T_{j,N} = 0, \qquad j = 1, 2,..., \infty, \tag{2.4}$$

where

$$\Delta_N = \sum_{l=1}^{\infty} b_l T_{l,N}^2 . \tag{2.5}$$

LEMMA 2.1. *If the initial conditions* (1.2) *satisfy the condition*

$$\sum_{j=1}^{\infty} \beta_j^2 < \infty \tag{2.6a}$$

$$\sum_{j=1}^{\infty} a_j\alpha_j^2 < \infty, \tag{2.6b}$$

then the function Δ_N is uniformly bounded independent of N.

Proof. In view of the fact that $T_{j,N} = 0$ for $j > N$, Eq. (2.2) may be rewritten in the form

$$\sum_{j=1}^{\infty} (T'_{j,N})^2 + \sum_{j=1}^{\infty} a_j T_{j,N}^2 + \frac{1}{2}\Big(\sum_{j=1}^{\infty} b_j T_{j,N}^2\Big)^2 \leqslant h, \tag{2.7}$$

where

$$h = \sum_{j=1}^{\infty} \beta_j^2 + \sum_{j=1}^{\infty} a_j\alpha_j^2 + \frac{1}{2}\Big(\sum_{j=1}^{\infty} b_j\alpha_j^2\Big)^2. \tag{2.8}$$

However, (2.6) combined with the assumption that $b_j \leqslant a_j/\mu$ [cf. (1.6)] show that h is finite and, hence, each of the sums in (2.7) are uniformly bounded, i.e.,

$$\sum_{j=1}^{\infty} (T'_{j,N})^2 \leqslant M_1 , \tag{2.9a}$$

$$\sum_{j=1}^{\infty} a_j T_{j,N}^2 \leqslant M_2 , \tag{2.9b}$$

$$\sum_{j=1}^{\infty} b_j T_{j,N}^2 \leqslant M_3 . \tag{2.9c}$$

The lemma follows from (2.9c). Q.E.D.

LEMMA 2.2. *If the initial conditions* (1.2) *satisfy the conditions* (2.6) *and*

$$\sum_{j=1}^{\infty} b_j \beta_j^2 < \infty, \tag{2.10a}$$

$$\sum_{j=1}^{\infty} a_j b_j \beta_j^2 < \infty, \tag{2.10b}$$

then there exists an interval $0 \leqslant t < t_c$ *such that* $| \Delta_N' |$ *is uniformly bounded independent of* N *on every closed subinterval* $0 \leqslant t \leqslant t^* < t_c$.

Proof. Define a function $E_{j,N}$ as (cf. [7])

$$E_{j,N} = (T'_{j,N})^2/(a_j + b_j \Delta_N) + T_{j,N}^2 \geqslant 0, \qquad j = 1, 2,..., \infty. \tag{2.11}$$

Equation (2.11) may be differentiated and combined with (2.4) to show

$$E'_{j,N} = \frac{-b_j \Delta_N'}{a_j + b_j \Delta_N} \left(\frac{(T'_{j,N})^2}{a_j + b_j \Delta_N} \right) \leqslant \frac{b_j \mid \Delta_N' \mid}{a_j + b_j \Delta_N} E_{j,N} \tag{2.12}$$

or, equivalently,

$$E_{j,N}(t) \leqslant E_{j,N}(0) \exp \left(\int_0^t \frac{b_j \mid \Delta_N' \mid}{a_j + b_j \Delta_N} \, d\tau \right). \tag{2.13}$$

It follows from (1.6) that

$$E_{j,N}(t) \leqslant E_{j,N}(0) \exp \left(\int_0^t \frac{\mid \Delta_N' \mid}{\mu + \Delta_N} \, d\tau \right). \tag{2.14}$$

Thus, the following bounds are obtained

$$T_{j,N}^2 \leqslant (\beta_j^2/ \rightarrow (a_j + b_j \Delta_N(0)) + \alpha_j^2) \exp \left(\int_0^t \frac{\mid \Delta_N' \mid}{\mu + \Delta_N} \, d\tau \right), \tag{2.15a}$$

$$(T'_{j,N})^2/(a_j + b_j \Delta_N) \leqslant (\beta_j^2/(a_j + b_j \Delta_N(0)) + \alpha_j^2) \exp \left(\int_0^t \frac{\mid \Delta_N' \mid}{\mu + \Delta_N} \, d\tau \right). \tag{2.15b}$$

The function Δ_N may be differentiated and the Schwarz inequality applied to show

$$| \Delta_N' | \leqslant 2 \sum_{l=1}^{\infty} b_l \, | T_{l,N} | \, | T'_{l,N} | \leqslant 2 \left\{ \sum_{l=1}^{\infty} b_l T_{l,N}^2 \sum_{l=1}^{\infty} b_l (T'_{l,N})^2 \right\}^{1/2}$$

$$\leqslant 2(\mu + \Delta_N)^{1/2} \left\{ \sum_{l=1}^{\infty} b_l (T'_{l,N})^2 \right\}^{1/2}. \tag{2.16}$$

The sum on the right of (2.16) may be estimated using (2.15b). Thus,

$$(T'_{j,N})^2 \leqslant \frac{a_j}{\mu} (\mu + \Delta_N)(\beta_j^2/(a_j + b_j \Delta_N(0)) + \alpha_j^2) \exp \left(\int_0^t \frac{| \Delta_N' |}{\mu + \Delta_N} \, d\tau \right) \tag{2.17}$$

or

$$\sum_{j=1}^{\infty} b_j (T'_{j,N})^2 \leqslant K^2 (\mu + \Delta_N) \exp \left(\int_0^t \frac{| \Delta_N' |}{\mu + \Delta_N} \, d\tau \right), \tag{2.18}$$

where

$$K^2 = \sum_{j=1}^{\infty} (b_j \beta_j^2 + a_j b_j \alpha_j^2)/\mu. \tag{2.19}$$

Note that the assumption (2.10) guarantees that K is finite. Combining (2.16) and (2.19) yields

$$| \Delta_N' | \leqslant 2K(\mu + \Delta_N) \exp \left(\frac{1}{2} \int_0^t \frac{| \Delta_N' |}{\mu + \Delta_N} \, d\tau \right) \tag{2.20}$$

or

$$\frac{-d}{dt} \exp \left(-\frac{1}{2} \int_0^t \frac{| \Delta_N' |}{\mu + \Delta_N} \, d\tau \right) \leqslant K. \tag{2.21}$$

The inequality (2.21) may be integrated to obtain

$$\exp \left(\frac{1}{2} \int_0^t \frac{| \Delta_N' |}{\mu + \Delta_N} \, d\tau \right) \leqslant 1/(1 - Kt) \tag{2.22}$$

for t in the interval

$$0 \leqslant t < t_c = 1/K. \tag{2.23}$$

Therefore, the exponential in (2.21) is uniformly bounded independent of N for t lying in any closed subinterval $0 \leqslant t \leqslant t^* < t_c$. This result in conjunction with (2.20) and (2.9c) yields

$$| \Delta_N' | \leqslant 2K(\mu + M_3)/(1 - Kt) \tag{2.24}$$

for $0 \leqslant t \leqslant t^* < t_c$. Q.E.D.

Lemma 2.2 only guarantees the boundedness of $|\Delta_N'|$ over a finite interval. However, it may be noted that $t_c \to \infty$ as $K \to 0$, i.e., the length of the interval over which $|\Delta_N'|$ is bounded grows arbitrarily large as the initial data shrinks to zero. It is possible to extend the preceding discussion to include the case $a_j = 0$ for $j = 1, 2,..., \infty$ ($\mu = 0$) if $\Delta_N(0) \neq 0$ when N is sufficiently large (cf. [1]). This latter condition has special significance in the theory of "extensible" strings, since it is equivalent to the requirement that the related partial differential equation should be hyperbolic at $t = 0$.

Lemmas 2.1 and 2.2 show that if the initial data satisfy the conditions (2.6) and (2.10), the functions Δ_N form a bounded equicontinuous sequence. Therefore, the Arzela-Ascoli lemma (cf. [6]) guarantees the existence of a subsequence Δ_{N_i} which converges uniformly to a continuous function Δ on the interval $0 \leqslant t \leqslant t^* < t_c$. Let T_j be the solution of the *linear* equation

$$T_j'' + (a_j + b_j\Delta)\, T_j = 0, \qquad j = 1, 2,..., \infty \tag{2.25}$$

satisfying the initial conditions (1.2). There is no difficulty in showing that $T_{j,N_i} \to T_j$ as $N_i \to \infty$.

LEMMA 2.3. The solutions T_j of (2.25) are solutions of the system of eqs. (1.1).

Proof. To show that the solutions of (2.25) are solutions of (1.1), it suffices to show that

$$\Delta = \sum_{l=1}^{\infty} b_l T_l^2. \tag{2.26}$$

For this purpose, write

$$\begin{aligned} \left|\Delta - \sum_{l=1}^{\infty} b_l T_l^2\right| &\leqslant |\Delta - \Delta_{N_i}| + \sum_{l=1}^{\infty} b_l\, |T_{l,N_i}^2 - T_l^2| \\ &\leqslant |\Delta - \Delta_{N_i}| + \sum_{l=1}^{M} b_l\, |T_{l,N_i}^2 - T_l^2| + \sum_{l=M+1}^{\infty} b_l (T_{l,N_i}^2 + T_l^2). \end{aligned} \tag{2.27}$$

In view of the fact that $T_{l,N_i} \to T_l$, the second sum on the right of (2.27) converges (cf. (2.15a)). Therefore, by first choosing M and then N_i, the right side of (2.27) can be made arbitrarily small. Q.E.D.

The results of this section can now be summarized.

THEOREM 2.1. *The system of Eqs.* (1.1) *with intial data* (1.2) *has a solution* on any closed subinterval $0 \leqslant t \leqslant t^* < t_c$ [*cf.* (2.23)] *if* a_j *and* b_j *satisfy condition* (1.6) *and the initial data satisfy conditions* (2.6) *and* (2.10).

It may be noted that if $b_j \to \infty$ as $j \to \infty$, the requirement that the initial conditions satisfy both (2.6) and (2.10) is redundant since (2.10) will imply (2.6). In any case, the fact that the functions T_j are the limits of T_{j,N_i} as $N_i \to \infty$ shows that [cf. (2.9)]

$$\sum_{j=1}^{\infty} (T_j')^2 \leqslant M_1 \tag{2.28a}$$

$$\sum_{j=1}^{\infty} a_j T_j^2 \leqslant M_2 \tag{2.28b}$$

$$\sum_{j=1}^{\infty} b_j T_j^2 \leqslant M_3 \tag{2.28c}$$

and, in addition, (2.15) shows that

$$\sum_{j=1}^{\infty} b_j (T_j')^2 \leqslant K^2(\mu + M_3)/(1 - Kt)^2, \tag{2.29a}$$

$$\sum_{j=1}^{\infty} a_j b_j T_j^2 \leqslant \mu K^2/(1 - Kt)^2, \tag{2.29b}$$

for $0 \leqslant t \leqslant t^* < t_c$, i.e., the infinite sums in (2.28) and (2.29) all converge. At the beginning of this section it was noted that Eq. (2.1) has a solution for all $t \geqslant 0$. This followed from the fact that it was possible to continue solutions of (2.1) for all $t \geqslant 0$. Thus, the question arises whether this is possible for solutions of the infinite system (1.1), i.e., do the solutions of (1.1) continue beyond $t = t_c$. If it were true that the sums in (2.29) converged at $t = t_c$, then the above existence proof could be reapplied at $t = t_c$ to show that the solution exists beyond t_c. However, it is not at all clear that this is the case. On the other hand, it does follow that if the solution does cease to exist for some value of t, the cause will not be the unbounded growth of the solution since $| T_j |$ and $| T_j' |$ remain bounded for all time [cf. (2.28)].

3. Uniqueness

In this section it will be shown that the solution of (1.1) satisfying the initial conditions (1.2) is unique. Specifically, it will be shown that if the initial data satisfies the conditions (2.6) and (2.10), then (1.1) has at most one solution satisfying the condition

$$\sum_{j=1}^{\infty} b_j(T_j')^2 < \infty, \tag{3.1a}$$

$$\sum_{j=1}^{\infty} a_j b_j T_j^2 < \infty. \tag{3.1b}$$

The condition (3.1) was certainly satisfied by the solutions which were found in Section 2. Thus, this result will show that it was not necessary to pick a subsequence in finding the function Δ (cf., Section 2) since the total sequence must converge to a unique limit. Therefore, the uniqueness result will make the existence proof of Section 2 constructive.

Let T_j and S_j be solutions of (1.1) satisfying the initial conditions (1.2), i.e., T_j is a solution of (2.25) where Δ is given by (2.26) and S_j is a solution of

$$S_j + (a_j + b_j\Gamma)\, S_j = 0, \qquad j = 1, 2,..., \infty, \tag{3.2}$$

where

$$\Gamma = \sum_{l=1}^{\infty} b_l S_l^2. \tag{3.3}$$

The condition (3.1) implies that Δ (and Γ) are differentiable. Thus,

$$| \Delta' | \leqslant 2 \sum_{l=1}^{\infty} b_l \mid T_l \mid \mid T_l' \mid \leqslant 2 \left\{ \sum_{l=1}^{\infty} (T_l')^2 \sum_{l=1}^{\infty} b_l^2 T_l^2 \right\}^{1/2}$$

$$\leqslant 2 \left\{ \sum_{l=1}^{\infty} (T_l')^2 \sum_{l=1}^{\infty} a_l b_l T_l^2 / \mu \right\}^{1/2}. \tag{3.4}$$

The second sum on the right of (3.4) converges because of (3.1b) and the first sum converges in view of the fact that [cf. (2.7)]

$$\sum_{j=1}^{\infty} (T_j')^2 + \sum_{j=1}^{\infty} a_j T_j^2 + \frac{1}{2} \left(\sum_{j=1}^{\infty} b_j T_j^2 \right)^2 = h, \tag{3.5}$$

where h is given by (2.8). Note also that there is no difficulty in repeating the proof of Lemma 2.2 to show that

$$\exp\left(\frac{1}{2} \int_0^t \frac{| \Delta' |}{\mu + \Delta}\, d\tau \right) \leqslant 1/(1 - Kt), \tag{3.6}$$

where K is given by (2.19) and $0 \leqslant t < t_c$. Define U_j as

$$U_j = T_j - S_j, \tag{3.7}$$

so that U_j is a solution of the ordinary differential equation

$$U_j'' + (a_j + b_j\Delta)\, U_j = b_j(\Gamma - \Delta)\, S_j\,, \tag{3.8}$$

and satisfies the initial conditions

$$U_j(0) = U_j'(0) = 0. \tag{3.9}$$

The object is to show that $U_j \equiv 0$ in some interval $0 \leqslant t \leqslant t_1 < t_c$. An obvious approach would be to try to find a Gronwall type inequality for U_j. However, the form of (3.8) makes it difficult to find such an inequality. Thus a slightly different approach will be employed.

Define a function E_j as

$$E_j = (U_j')^2/(a_j + b_j\Delta) + U_j^2 \geqslant 0. \tag{3.10}$$

Equation (3.10) may be differentiated and (3.8) used to show that

$$\begin{aligned} E_j' &= \frac{b_j\Delta'}{a_j + b_j\Delta}\left(\frac{(U_j')^2}{a_j + b_j\Delta}\right) + \frac{2b_j(\Gamma - \Delta)}{a_j + b_j\Delta}\, U_j' S_j \\ &\leqslant \frac{|\Delta'|}{\mu + \Delta} E_j + 2\,\frac{b_j\,|\Gamma - \Delta|}{a_j + b_j\Delta}\,|U_j'|\,|S_j| \end{aligned} \tag{3.11}$$

or

$$\frac{d}{dt} E_j \exp\left(-\int_0^t \frac{|\Delta'|}{\mu + \Delta}\, d\tau\right) \leqslant 2\,\frac{b_j\,|\Gamma - \Delta|}{a_j + b_j\Delta}\,|U_j'|\,|S_j|. \tag{3.12}$$

After integration, (3.12) becomes

$$E_j \leqslant 2 \exp\left(\int_0^t \frac{|\Delta'|}{\mu + \Delta}\, d\tau\right) \int_0^t \frac{b_j\,|\Gamma - \Delta|}{a_j + b_j\Delta}\,|U_j'|\,|S_j|\, d\tau \tag{3.13}$$

or

$$\sum_{j=1}^{\infty} E_j \leqslant \frac{2}{(1 - Kt)^2} \int_0^t |\Gamma - \Delta| \sum_{j=1}^{\infty} \frac{b_j\,|U_j'|\,|S_j|}{a_j + b_j\Delta}\, d\tau. \tag{3.14}$$

The quantities occurring in the integrand of (3.14) may be estimated. Thus,

$$\begin{aligned} \sum_{j=1}^{\infty} \frac{b_j\,|U_j'|\,|S_j|}{a_j + b_j\Delta} &\leqslant \left\{\sum_{j=1}^{\infty} (U_j')^2/(a_j + b_j\Delta) \sum_{j=1}^{\infty} b_j^2 S_j^2/(a_j + b_j\Delta)\right\}^{1/2} \\ &\leqslant \left\{\sum_{j=1}^{\infty} (U_j')^2/(a_j + b_j\Delta) \sum_{j=1}^{\infty} b_j S_j^2/\mu\right\}^{1/2} \end{aligned} \tag{3.15}$$

and

$$|\Gamma - \Delta| \leqslant \sum_{j=1}^{\infty} b_j |S_j + T_j| |U_j| \leqslant \left\{\sum_{j=1}^{\infty} U_j^2 \sum_{j=1}^{\infty} b_j^2(S_j + T_j)^2\right\}^{1/2}$$

$$\leqslant \left\{\sum_{j=1}^{\infty} U_j^2 \sum_{j=1}^{\infty} a_j b_j (S_j + T_j)^2/\mu\right\}^{1/2}. \tag{3.16}$$

The inequalities (3.15) and (3.16) may be combined with (3.14) to show

$$\sum_{j=1}^{\infty} E_j \leqslant \frac{2}{(1-Kt)^2} \int_0^t G(t) \left\{\sum_{j=1}^{\infty} (U_j')^2/(a_j + b_j\Delta) \sum_{j=1}^{\infty} U_j^2\right\}^{1/2} d\tau, \tag{3.17}$$

where

$$G(t) = \left\{\sum_{j=1}^{\infty} b_j S_j^2 \sum_{j=1}^{\infty} a_j b_j (T_j + S_j)^2\right\}^{1/2} \Big/ \mu. \tag{3.18}$$

The function $G(t)$ is a bounded function on any closed subinterval $0 \leqslant t \leqslant t^* < t_c$ [cf. (3.1) and (3.5)]. Since $G(t)$ is bounded, it is clear that there exists an interval $0 \leqslant t \leqslant t_1 < t_c$ such that

$$G(t)\, t/(1 - Kt)^2 < 1, \tag{3.19}$$

for all t in the interval $0 \leqslant t \leqslant t_1 < t_c$.

LEMMA 3.1. *The integrand which occurs in* (3.17) *vansishes identically for t in the interval* $0 \leqslant t \leqslant t_1$.

Proof. Assume the integrand does not vanish identically. In this case, the integrand has a positive maximum in the interval $0 \leqslant t \leqslant t_1$ at some point $t = \eta$ ($\eta \neq 0$ since the integrand vanishes there). The inequality (3.17) holds throughout $0 \leqslant t < t_c$ and, consequently, it holds at $t = \eta$. Therefore, (3.17) may be estimated at $t = \eta$ by

$$\sum_{j=1}^{\infty} E_j(\eta) \leqslant 2\,\frac{G(\eta)\eta}{(1-K\eta)^2} \left\{\sum_{j=1}^{\infty} (U_j')^2/(a_j + b_j\Delta) \sum_{j=1}^{\infty} U_j^2\right\}^{1/2}_{t=}$$

$$< 2 \left\{\sum_{j=1}^{\infty} (U_j')^2/(a_j + b_j\Delta) \sum_{j=1}^{\infty} U_j^2\right\}^{1/2}_{t=\eta}. \tag{3.20}$$

Recalling the definition of E_j, the inequality (3.20) may be rewritten

$$\left\{\left(\sum_{j=1}^{\infty} (U_j')^2/(a_j + b_j\Delta)\right)^{1/2} - \left(\sum_{j=1}^{\infty} U_j^2\right)^{1/2}\right\}^2_{t=\eta} < 0. \tag{3.21}$$

This is the desired contradiction. Q.E.D.

Lemma 3.1, in conjunction with (3.17), shows that $E_j \equiv 0$ for all t in the interval $0 \leqslant t \leqslant t_1$ or, equivalently, $T_j \equiv S_j$ in this interval. However, this result is easily extended to show that $S_j \equiv T_j$ as long as (3.1) is satisfied. Assume there exists a set of points for which $S_j \equiv T_j$ and for which S_j and T_j satisfy condition (3.1). This set has a greatest lower bound $t' \geqslant t_1$. The functions $S_j = T_j$ at t' and (3.1) and (3.5) will hold there. Therefore, by repeating the above procedure at $t = t'$, it may be shown that the solution is unique to the right of t'. Therefore, t' cannot be the greatest lower bound of points for which $T_j \neq S_j$.

THEOREM 3.1. *The system of Eqs.* (1.1) *with initial data* (1.2) *has at most one solution satisfying conditions* (3.1) *if the initial data satisfy conditions* (2.6) *and* (2.10).

REFERENCES

1. R. W. DICKEY, Infinite Systems of Nonlinear Oscillation Equations Related to the String. *Proc. of the Amer. Math. Soc.* **23** (1969), 459–468.
2. R. W. DICKEY, Free Vibration and Dynamic Buckling of the Extensible Beam. *J. Math. Anal. Appl.* **29** (1970), 443–454.
3. J. J. STOKER, "Nonlinear Vibrations in Mechanical and Electrical Systems." Interscience, New York, 1950.
4. S. WOINOWSKY-KRIEGER, The effect of axial force on the vibration of hinged bars. *J. Appl. Mech.* **17** (1950), 35–36.
5. J. G. EISLEY, Nonlinear vibrations of beams and rectangular plates. *Z. Agnew. Math. Phys.* **15** (1964), 167–175.
6. E. A. CODDINGTON AND N. LEVINSON, "Theory of Ordinary Differential Equations." McGraw-Hill, New York, 1955.
7. J. S. W. WONG, Explicit bounds for solutions of certain second order nonlinear differential equations. *J. Math. Anal. Appl.* **17** (1967), 339–342.